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PERIODIC AND ALMOST PERIODIC TRAJECTORIES IN CISLUNAR SPACE

Summary Report

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Introduction

The object of this study is to investigate the finite time stability of periodic and almost periodic trajectories in cislunar space. To this end, we have been pursuing a program of research in the application of normal forms for Hamiltonian systems. When completed, this research will hopefully provide us with the analytical and computational tools which will enable us to treat a wide variety of mathematical models.

This report will be divided into three parts. The first part will discuss the concept of normal forms and their application. This provides the theoretical basis for work done under this contract. The second part will discuss the computer package, now in its final stages of development, which facilitates the application of the concept of normal forms to problems in celestial mechanics. This package, among other things, enables us to find approximate analytical expressions for trajectories which are near points of equilibrium of Hamiltonian systems. The third part discusses the analysis necessary, in conjunction with the computer output, to enable us to make the desired statements about finite time stability of actual trajectories and the accuracy of the approximate analytical expressions.

I. Normal Forms for Hamiltonian Systems

The theory of Normal Forms for Hamiltonian Systems was introduced by Birkhoff[1]. It is this theory which we wish to exploit in order to obtain statments on finite time stability. We start by assuming that we wish to solve Hamilton's equations

$$\dot{x}_i = \partial H / \partial y_i \ ,$$

$$\dot{y}_i = -\partial H / \partial x_i \ , \qquad i = 1, 2, \ldots n, \tag{1}$$

with a Hamiltonian $H(x_1, \ldots x_n, y_1, \ldots, y_n)$ which is independent of time and has the form:

$$H(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} (\omega_i / 2)(x_i^2 + y_i^2) + H_3 + H_4 + \ldots + H_s + \ldots \tag{2}$$

where H_s is a homogeneous polynomial of degree s . The implication of this assumption as it pertains to specific problems will be discussed later. The ω_i's are assumed to be real. We now define what we mean by a normal form for Hamiltonian systems. We say that a Hamiltonian function $\Gamma(\xi_1, \ldots, \xi_n, \eta_1, \ldots \eta_n)$ is in normal form up to degree s , if

$$\Gamma = \Gamma' + \Gamma'' \tag{3}$$

where Γ' is a polynomial of degree $\leq s/2$ in the variables $\zeta_i = \frac{1}{2}(\xi_i^2 + \eta_i^2)$

and Γ'' is a power series whose terms all have degree at least $s + 1$ in the variables, ξ_i , η_i . The process of normalizing a Hamiltonian consists of making a series of $(s-2)$ canonical transformations such that the Hamiltonian (2) expressed in the new variables is in normal form up to degree s .

Suppose then we have the equations

$$\dot{\xi}_i = \partial \Gamma / \partial \eta_i \ ,$$

$$\dot{\eta}_i = -\partial \Gamma / \partial \xi_i \ , \qquad i = 1, 2, \ldots n, \tag{4}$$

[1] G.D. Birkhoff, Dynamical Systems, American Mathematical Society Colloquium Publications, Vol. 9 (1927) p. 74

where Γ is normalized up to order $s + 1$.

We first note that if Γ is replaced by Γ' the equations can be immediately integrated. To see this, we note that since Γ' has the form assumed above we have

$$\partial \Gamma'/\partial \eta_i = (\partial \Gamma'/\partial \zeta_i)\eta_i , \quad \partial \Gamma'/\partial \xi_i = (\partial \Gamma'/\partial \zeta_i)\xi_i .$$

Thus in this case Hamilton's equations become

$$\dot{\xi}_i = (\partial \Gamma'/\partial \zeta_i)\eta_i ,$$

$$\dot{\eta}_i = -(\partial \Gamma'/\partial \zeta_i)\xi_i , \quad i = 1,2,...n. \tag{5}$$

If we now multiply the first equation (5) by ξ_i and the second equation (5) by η_i, add and integrate, we see that ζ_i = constant; hence $\widetilde{\omega}_i = \partial \Gamma'/\partial \zeta_i$ is a constant (which depends on the initial values of $\zeta_1,...\zeta_n$.). Thus we may write down the solution to (5),

$$\xi_i = \xi_{io} \cos \widetilde{\omega}_i (t - t_o) + \eta_{io} \sin \widetilde{\omega}_i (t - t_o),$$

$$\eta_i = -\xi_{io} \sin \widetilde{\omega}_i (t - t_o) + \eta_{io} \cos \widetilde{\omega}_i (t - t_o), \quad i = 1,2,...n. \tag{6}$$

We shall see that these expressions can be used as approximations to the trajectories of the system (4). We see that they are almost periodic, (for $\widetilde{\omega}_i$ incommensurable), hence bounded. Now consider the full equations (4) which in view of the form of Γ can be written

$$\dot{\xi}_i = \frac{\partial \Gamma'}{\partial \zeta_i} \eta_i + \frac{\partial \Gamma''}{\partial \eta_i} ,$$

$$\dot{\eta}_i = -\frac{\partial \Gamma'}{\partial \zeta_i} \xi_i - \frac{\partial \Gamma''}{\partial \xi_i} , \quad i = 1,2,...n. \tag{7}$$

Now, if we let $u^2 = \sum_{i=1}^{n} \zeta_i$, then by assumption we can find C such that:

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$$\left| \frac{\partial \Gamma''}{\partial \eta_i} \right|, \quad \left| \frac{\partial \Gamma''}{\partial \xi_i} \right| \leq C \, u^{s+1} \quad \text{for } u \leq u_1 \text{ say.} \qquad (8)$$

Hence, if we multiply the first equation (7) by ξ_i, the second by η_i, add and sum over i we find

$$\left| u \frac{du}{dt} \right| \leq n C u^{s+2}.$$

From this we easily find that if u_o is the initial value of u we have

$$\left| \frac{1}{u_o^s} - \frac{1}{u^s} \right| \leq ns C \left| t - t_o \right| \quad \text{for } u, u_o \leq u_1.$$

We now ask how long it can take for u to double in value (if $2u_o \leq u_1$). For this value of t, say t_1 we have

$$\left| \frac{1}{u_o^s} - \frac{1}{2^s u_o^s} \right| \leq n s C \left| t_1 - t_o \right| ;$$

hence

$$\left| t_1 - t_o \right| \geq 1/2 \, ns C \, u_o^s \quad \text{if } 2 u_o \leq u_1. \qquad (9)$$

This is what we mean by finite time stability. We see that u can be considered as a measure of the distance of a point on the trajectory to the origin in the (ξ, η) space. Then equation (9) says that if u_o is sufficiently small then the time which must elapse before this "distance" can double in value is of the s-th order in reciprocal distance.

Now let initial values be given for ξ_i and η_i. We wish to compare the actual solution of (4) with the solutions, (6) of the approximate equations (5). Corresponding to the quantities ζ_{io}, η_{io} we have the initial values ζ_{io} and u_o. We assume now that $2u_o \leq u_1$ so that we may use (8). Then we have for $\left| t - t_o \right| \leq 1/2 ns C u_o^s$,

$$\left| \frac{\partial \zeta_i}{\partial t} \right| \leq 2^{s+2} C u_o^{s+2}.$$

Thus

$$\left| \zeta_i - \zeta_{io} \right| \leq 2^{s+2} C u_o^{s+2} \left| t - t_o \right|, \quad i = 1, \ldots n.$$

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Since Γ' is a polynomial, we can find a P such that

$$\left|\frac{\partial \Gamma'}{\partial \zeta_i} - \left(\frac{\partial \Gamma'}{\partial \zeta_i}\right)_o\right| \le P\sum_{i=1}^{n}\left|\zeta_i - \zeta_{io}\right| \le 2^{s+3}nCPu_o^{s+2}\left|t - t_o\right|, \quad i = 1,\ldots n.$$

Further we have from (7)

$$\left|\frac{d\xi_i}{dt} - \frac{\partial \Gamma'}{\partial \zeta_i}\eta_i\right|, \quad \left|\frac{d\eta_i}{dt} + \frac{\partial \Gamma'}{\partial \zeta_i}\xi_i\right| \le 2^{s+1}Cu_o^{s+1}, \quad i = 1,\ldots n;$$

thus combining the last 2 inequalities we conclude $(\widetilde{\omega}_i = (\partial \Gamma'/\partial \zeta_i)_o)$

$$\left|\frac{d\xi_i}{dt} - \eta_i\widetilde{\omega}_i\right|, \quad \left|\frac{d\eta_i}{dt} + \xi_i\widetilde{\omega}_i\right| \le 2^{s+1}Cu_o^{s+1} + 2^{s+4}nCPu_o^{s+3}\left|t - t_o\right|.$$

For convenience denote the right hand side of the last inequality by F; then

$$\left|\cos\widetilde{\omega}_i t\left(\frac{d\xi_i}{dt} - \eta_i\widetilde{\omega}_i\right) - \sin\widetilde{\omega}_i t\left(\frac{d\eta_i}{dt} + \xi_i\widetilde{\omega}_i\right)\right| \le 2F,$$

$$\left|\frac{d}{dt}\left(\xi_i\cos\widetilde{\omega}_i t - \eta_i\sin\widetilde{\omega}_i t\right)\right| \le 2F.$$

Similarly,

$$\left|\frac{d}{dt}\left(\xi_i\sin\widetilde{\omega}_i t + \eta_i\cos\widetilde{\omega}_i t\right)\right| \le 2F.$$

Therefore,

$$\left|\left(\xi_i\cos\widetilde{\omega}_i t - \eta_i\sin\widetilde{\omega}_i t\right) - \left(\xi_{io}\cos\widetilde{\omega}_i t_o - \eta_{io}\sin\widetilde{\omega}_i t_o\right)\right| \le 2\int_{t_o}^{t}Fdt,$$

$$\left|\left(\xi_i\sin\widetilde{\omega}_i t + \eta_i\cos\widetilde{\omega}_i t\right) - \left(\xi_{io}\sin\widetilde{\omega}_i t_o + \eta_{io}\cos\widetilde{\omega}_i t_o\right)\right| \le 2\int_{t_o}^{t}Fdt;$$

thus

$$\left|\xi_i - \left[\xi_{io}\cos\widetilde{\omega}_i(t-t_o) + \eta_{io}\sin\widetilde{\omega}_i(t-t_o)\right]\right| \le 4\int_{t_o}^{t}Fdt,$$

$$\left| \eta_i - [-\xi_{io} \sin \widetilde{\omega}_i (t - t_o) + \eta_{io} \cos \widetilde{\omega}_i (t - t_o)] \right| \leq 4 \int_{t_o}^{t} F \, dt,$$

or finally

$$\left| \xi_i - [\xi_{io} \cos \widetilde{\omega}_i (t - t_o) + \eta_{io} \sin \widetilde{\omega}_i (t - t_o)] \right|,$$

$$\left| \eta_i - [-\xi_{io} \sin \widetilde{\omega}_i (t - t_o) + \eta_{io} \cos \widetilde{\omega}_i (t - t_o)] \right|, \tag{10}$$

$$\leq 2^{s+3} C u_o^{s+1} |t - t_o| + 2^{s+5} n C P u_o^{s+3} |t - t_o|^2, \quad i = 1, \ldots n.$$

Equation (10) gives an estimate for the error made in approximating the actual trajectories with the expressions given by (6) which are solutions of the simplified equations (5). This estimate is valid over the time interval $|t - t_o| \leq \dfrac{1}{2 n s C u_o^s}$.

The above analysis is essentially the same as that given by Birkhoff in reference 1. In order to be able to exploit this analysis, three steps are necessary. First the canonical transformations which normalize the Hamiltonian and the normalized Hamiltonian must be computed. How this is done using the computer package will be discussed in the next section. Once this has been done the approximate solutions, (6) can easily be computed. Next we must make explicit the estimate (8) which we need to compute (9) which is our basic result. Finally it is necessary to translate this information back to the original coordinates. These last two steps will be discussed in part three.

II   Computer Program for Normalizing the Hamiltonian

As mentioned in Section I, the normalization process is one of applying a series of trans-formations, each of which normalizes the Hamiltonian to one higher degree. The basic result states that the Hamiltonian, (2) can be normalized up to degree N if

$$\sum_{i=1}^{n} \omega_i k_i \neq 0 \text{ for integers } k \text{ with } 0 < \Sigma |k_i| \leq N. \tag{11}$$

We will now show how this is done.

We assume that H has been normalized up to degree $s - 1$ and condition (11) is satis-fied for $N = s$. We introduce a canonical transformation from the old variables (x, y) to new variables (ξ, η) generated by a generating function $W^{(s)}(x_1, \ldots, x_n, \eta_1, \ldots, \eta_n)$, a homogeneous polynomial of degree s. The transformation is given (implicitly) by

$$\xi_i = x_i + \frac{\partial W^{(s)}}{\partial \eta_i},$$

$$y_i = \eta_i + \frac{\partial W^{(s)}}{\partial x_i}. \tag{12}$$

The idea is to choose $W^{(s)}$ (that is, the coefficients of the polynomial) in such a way that as many s-th order terms as possible are eliminated in the new Hamiltonian. If $\Gamma(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)$ is the new Hamiltonian we have

$$H(x_1, \ldots x_n, \eta_1 + \frac{\partial W^{(s)}}{\partial x_1}, \ldots, \eta_n + \frac{\partial W^{(s)}}{\partial x_n})$$

$$= \Gamma(x_1 + \frac{\partial W^{(s)}}{\partial \eta_1}, \ldots, x_n + \frac{\partial W^{(s)}}{\partial \eta_n}, \eta_1, \ldots, \eta_n).$$

We expand both sides in taylor series and find

$$H(x_1, \ldots x_n, \eta_1 \ldots \eta_n) + \sum_{i=1}^{n} \frac{\partial H}{\partial \eta_i} \frac{\partial W^{(s)}}{\partial x_i} + \frac{1}{2!} \sum_{\substack{i=1 \\ j=1}}^{n} \frac{\partial^2 H}{\partial \eta_i \partial \eta_j} \frac{\partial W^{(s)}}{\partial x_i} \frac{\partial W^{(s)}}{\partial x_j} + \ldots$$

$$\tag{13}$$

$$= \Gamma(x_1, \ldots x_n, \eta_1 \ldots \eta_n) + \sum_{i=1}^{n} \frac{\partial \Gamma}{\partial x_i} \frac{\partial W^{(s)}}{\partial \eta_i} + \frac{1}{2!} \sum_{\substack{i=1 \\ j=1}}^{n} \frac{\partial^2 \Gamma}{\partial x_i \partial x_j} \frac{\partial W^{(s)}}{\partial \eta_i} \frac{\partial W^{(s)}}{\partial \eta_j} + \ldots$$

We now write H and Γ as sums of homogeneous polynomials:

$$H = \sum_{n=2}^{\infty} H^{(n)} \, , \quad \Gamma = \sum_{n=2}^{\infty} \Gamma^{(n)} \, .$$

We equate terms of like degree in (13) and find

$$H^{(i)}(x_1, \ldots, x_n, \eta_1, \ldots, \eta_n) = \Gamma^{(i)}(x_1, \ldots, x_n, \eta_1, \ldots, \eta_n), \quad i=1,2,\ldots s-1,$$

so that the new Hamiltonian is normalized up to order $s-1$ since it was assumed that H was. Equating terms of degree s in (13), we have

$$H^{(s)} + \sum_{i=1}^{n} \frac{\partial H^{(2)}}{\partial \eta_i} \frac{\partial W^{(s)}}{\partial x_i} = \Gamma^{(s)} + \sum_{i=1}^{n} \frac{\partial \Gamma^{(2)}}{\partial x_i} \frac{\partial W^{(s)}}{\partial \eta_i} \quad .$$

In view of (2), this can be written

$$\sum_{i=1}^{n} \omega_i \left(\eta_i \frac{\partial W^{(s)}}{\partial x_i} - x_i \frac{\partial W^{(s)}}{\partial \eta_i} \right) = \Gamma^{(s)} - H^{(s)} \cdot \tag{14}$$

When coefficients of like terms are compared in the last equation, we obtain a system of linear equations in the coefficients of $W^{(s)}$ with unspecified non-homogeneous terms since $\Gamma^{(s)}$ is not known. It turns out that for s odd, the matrix of this system is non-singular (at this point condition (11) is used) so that the system can be solved with an arbitrary non-homogeneous term, i.e. we can find a $W^{(s)}$ so that (14) is satisfied with $\Gamma^{(s)} \equiv 0$. For even s the matrix does turn out to be singular. In this case, we can determine $W^{(s)}$ so that (14) is satisfied with $\Gamma^{(s)}$ a polynomial of degree $s/2$ in the variables $(x_i^2 + \eta_i^2)$ with known coefficients. Thus the new Hamiltonian is normalized up to order s. The higher order terms in Γ are then found by coefficient comparison in (13), since now $W^{(s)}$ is known.

Once the Hamiltonian has been normalized, we may make use of the theory presented in Section 1. Of course in order to use any information obtained from the analysis it is necessary to invert the successive transformations and compose them in order to express the new coordinates in terms of the old and vice-versa. In actual practice all the power series must be broken off at some point so that Γ'' in (3) cannot be written down exactly

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and the transformation from old to new coordinates can only be given approximately in explicit form.

Although the normalization process is conceptually simple, the amount of algebra involved is immense. For this reason, it was felt that it would be desirable to develop a computer program which would do the job of computing the transformation of coordinates and the normalized Hamiltonian. An attempt was made to use IBM's FORMAC system of non-numeric programming. This attempt proved to be only partially successful as it was found that during the execution of the program we developed the core storage capacity of the 7094 was exceeded. (As we found later, this is a constant source of trouble to the users of the experimental FORMAC compiler.) It was eventually decided to terminate the FORMAC effort and instead, use a program developed by Dr. Fred Gustavson at IBM. This program, written in FORTRAN IV computes the successive generating functions and the new Hamiltonian. A few of the unique features of this program should be mentioned here.

Most of the work involves multiplication, addition and differentiation of polynomials in $2n$ indeterminants. To do this a multi-index mapping system is developed which enables the program to store the coefficients of a polynomial in a one dimensional array. One subroutine is provided which multiplies two polynomials and stores the coefficients of the product polynomial in the proper sequence. Another subroutine performs differentiation of polynomials. In computing the new Hamiltonian (by use of (13)) a great deal of bookkeeping must be done to collect all terms of a given degree. Hence, the logical structure of this program is quite complex.

Our computer package consists of three parts. The first is the IBM program discussed above. The other two parts have been developed here at General Precision. The second part of the package is a program which performs the calculations to obtain the explicit relationship between the old and new coordinates by constructing power series from equation (12). This program gives the relation between old and new coordinate systems as (truncated) power series. This program uses many of the ideas developed in the IBM program.

A third program solves equation (12) numerically to give point-by-point transformations from one set of coordinates to another. This program will be used to check the second program and also to provide important information about the transformations themselves. This is important in the analysis as discussed in the next section.

III   Analytical Problems

In order to apply the concept of normal forms to specific problems certain analytical considerations must be made. These will now be discussed.

We assumed that the Hamiltonian has the form (2). Of course in actual problems some preliminary transformations must be made in order to bring it into this form. In general, we must first make a translation in position-momentum space to bring the equilibrium point to the origin. The special form of H_2 results from applying a linear canonical transformation. The statement that the ω_i's are all real is equivalent to the condition that the eigenvalues of the linearized Hamiltonian equations are pure imaginary. In studying trajectories of the restricted three-body problem in the neighborhood of the triangular libration points for instance, we know that this is true for the earth-moon mass ratio. Of course any constant term in the Hamiltonian is discarded as it plays no role in the equations.

In order to obtain the results we are interested in, the estimate (8) must be made explicit. Furthermore, the ranges and domains of the transformations (12) must be known in order to refer our conclusions about the finite time stability back to the original set of coordinates. Notice that Γ'' is not given exactly by the computer since the computer produces only truncated series. An analysis has been made which enables us to obtain estimates for C and u_1 in equation (8) and estimates for the ranges and domains of the transformation (12) but it was found that these estimates were too crude to be of practical value when applied to specific problems. Thus, it has been concluded that in order to obtain useful results from this method a careful numerical study must be made of the transformations. This will be done using the third program discussed in Section 2. Only when the ranges and domains of the transformations are accurately known can the method be expected to yield useful information.

Summary

The work performed under this contract represents an endeavor to set up an apparatus for studying finite time stability in a wide variety of models. At first, we have studied only autonomous systems in which we are interested in finite time stability of equilibrium points. As a first application we are applying our results to the problem of trajectories of the re-stricted three-body problem near the triangular libration points in two and three dimensions. In later work, we hope to extend our results to non-autonomous systems in order to study stability of periodic orbits in the restricted three-body problem and the stability of libration points in the elliptic three-body problem. In this case although many of the details are as presented above, the computation of the normalized Hamiltonian becomes more difficult because all coefficients become functions of time.